MULTISETS as a TOOL in DECISION MAKING

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Introduction

Fuzzy sets:

- introduced by Lotfi A. Zadeh in 1965
- a way to deal with real-life situations where there is either limited knowledge or some sort of implicit ambiguity
- the membership degree is a value in the interval $[0, 1]$

Extensions:

- it could be paradoxical that the membership value itself should be one precise real number
- different generalizations appeared as a way to solve this paradox
- in the case the membership value is an arbitrary subsets of $[0, 1]$, the generalization is called hesitant fuzzy sets (Torra, 2010)
- they are introduced as an interesting tool for decision making (Xia & Xu, 2011), but they have important drawbacks

We aim to...

- define a more appropriate generalization of the fuzzy sets, useful in this environment

In particular...

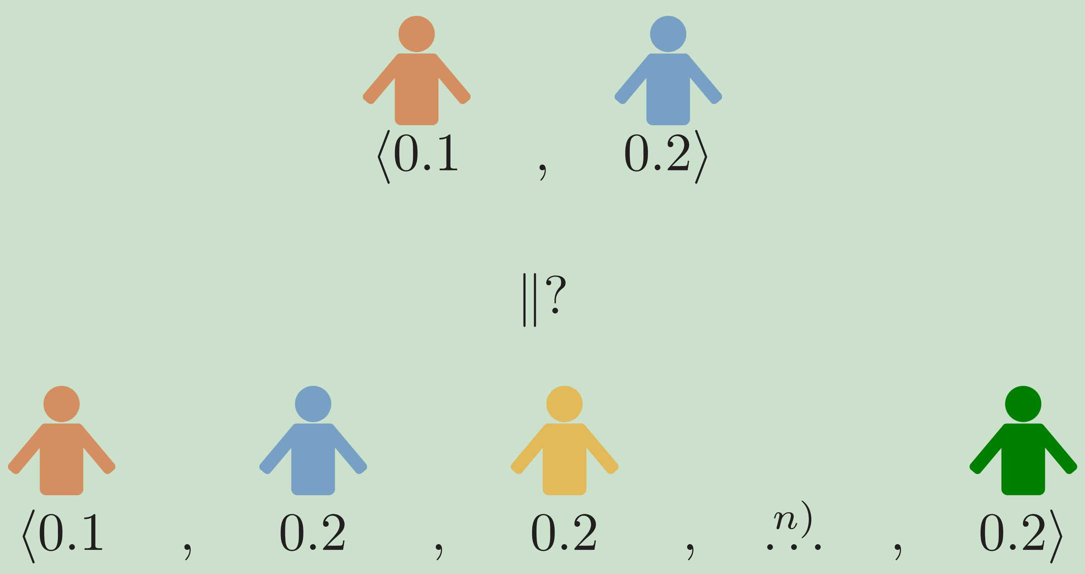
1. Define fuzzy multisets
2. Define complement, union and intersection of fuzzy multisets
3. Study their properties

Main drawbacks for hesitant fuzzy sets

For hesitant fuzzy sets the order of the elements in the set is not important and moreover, the repetition are not allowed. However,

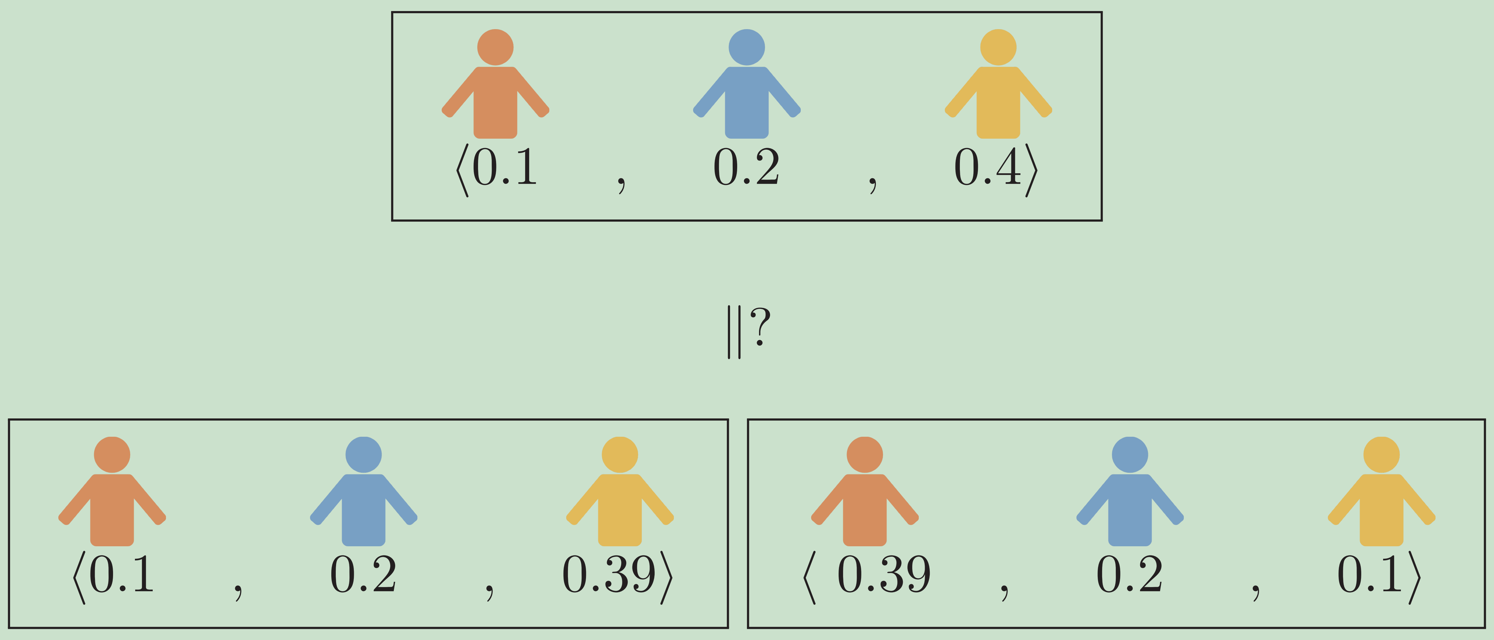
- **The repetitions are important!!!**

  It is not the same if one expert is considering a membership value than if this is considered by one hundred experts.



- **The order is important!!!**

  It is not the same if the opinion of the same expert is similar or different.



The solution: fuzzy multisets

- **Fuzzy multiset**: A fuzzy multiset $\hat{A}$ over X (universe) is characterized by a function $\hat{A}\colon X \to \mathbb{N}^{[0,1]}$. The family of all the fuzzy multisets over X is called the fuzzy power multiset over X and is denoted by $\mathscr{FM}(X)$.

  Example: Say we have a single-element universe $X = \{x\}$. We can define a fuzzy multiset $\hat{A}$ as $\hat{A}(x) = \langle 0.1, 0.2, 0.2\rangle$ or by means the map $Count_{\hat{A}(x)}\colon [0,1] \to \mathbb{N}$ defined as $Count_{\hat{A}(x)}(0.1) = 1$, $Count_{\hat{A}(x)}(0.2) = 2$ and $Count_{\hat{A}(x)}(t) = 0$ for any $t \neq 0.1$ and $t \neq 0.2$.

- **Complement**: Let $\hat{A} \in \mathscr{FM}(X)$ be a fuzzy multiset. The complement of $\hat{A}$ is the fuzzy multiset $\hat{A}^c$ defined by the following count function: $Count_{\hat{A}^c(x)}(t) = Count_{\hat{A}(x)}(1-t), \quad \forall x \in X, \quad \forall t \in [0,1]$.

  Example: If we have a two-element universe $X = \{x, y\}$, then a fuzzy multiset $\hat{A}$ with $\hat{A}(x) = \langle 0.3\rangle$ and $\hat{A}(y) = \langle 0.5, 0.8, 0.8\rangle$ has the complement $\hat{A}^c(x) = \langle 0.7\rangle$ and $\hat{A}^c(y) = \langle 0.5, 0.2, 0.2\rangle$.

- **Intersection and union**: Let $\hat{A}, \hat{B} \in \mathscr{FM}(X)$ be two fuzzy multisets. The aggregated intersection (union) of $\hat{A}$ and $\hat{B}$ is a fuzzy multiset $\hat{A} \cap^a \hat{B}$ such that for any element $x \in X$, $\hat{A} \cap^a \hat{B}(x)$ is the union, in the crisp multiset sense, of the regularised (s_A, s_B)-ordered intersections (unions) for all the possible pairs of ordering strategies (s_A, s_B), that is,

$$\hat{A} \cap^a \hat{B}(x) = \bigcup_{\substack{s_{\hat{A}} \in \mathcal{OS}(\hat{A}^r)\\ s_{\hat{B}} \in \mathcal{OS}(\hat{B}^r)}} \hat{A} \cap^r_{(s_A, s_B)} \hat{B}(x), \quad \text{and} \quad \hat{A} \cup^a \hat{B}(x) = \bigcup_{\substack{s_{\hat{A}} \in \mathcal{OS}(\hat{A}^r)\\ s_{\hat{B}} \in \mathcal{OS}(\hat{B}^r)}} \hat{A} \cup^r_{(s_A, s_B)} \hat{B}(x), \quad \forall x \in X.$$

  Example: For two fuzzy multisets $\hat{E}(x) = \langle 0.1, 0.4\rangle$ and $\hat{F}(x) = \langle 0.2, 0.3\rangle$. There are two possible ordering strategies for $\hat{E}$, resulting in the sequences $(0.1, 0.4)$ and $(0.4, 0.1)$, and two possible ordering strategies for $\hat{F}$, resulting in the sequences $(0.2, 0.3)$ and $(0.3, 0.2)$. This leads to the four sequences of pairwise minima, $(0.1, 0.3)$, $(0.1, 0.2)$, $(0.2, 0.1)$, $(0.3, 0.1)$, which result in two ordered intersections, $\langle 0.1, 0.3\rangle$, $\langle 0.1, 0.2\rangle$; and to the four sequences of pairwise maxima, $(0.2, 0.4)$, $(0.3, 0.4)$, $(0.4, 0.3)$, $(0.4, 0.2)$, which result in two ordered unions, $\langle 0.2, 0.4\rangle$, $\langle 0.3, 0.4\rangle$. By taking the union, in the crisp multiset sense, we get the aggregated intersection and union: $\hat{E} \cap^a \hat{F}(x) = \langle 0.1, 0.2, 0.3\rangle$ and $\hat{E} \cup^a \hat{F}(x) = \langle 0.2, 0.3, 0.4\rangle$.

Conclusions and references

At a glance

- Study in detail of the main operations between fuzzy multisets.
- Coherent definitions from different points of view.
- Fuzzy multisets are very appropriate tools to deal with real applications.

References

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[3] L.A. Zadeh, Fuzzy Sets, Information Control 8 (3 (1965) 338–353.